THE LABELLED IMMERSION CODE
FOR VIRTUAL KNOT DIAGRAMS

A **labelled immersion code** is a descriptive syntax for a virtual knot diagram and *a fortiori* a classical knot diagram. As the name suggests it is based on a description of the immersion of the diagram in the plane, or implicitly on a two dimensional sphere.

1 Immersion codes

Let $\mathcal{I}(D)$ be the immersion in the plane of an oriented virtual knot diagram D with n crossings, where n includes both real and virtual crossings. Thus $\mathcal{I}(D)$ is a 4-regular plane graph with n vertices and $2n$ edges, and it inherits an orientation from D.

Number the edges in $\mathcal{I}(D)$ consecutively $0, \ldots, 2n-1$, starting at any edge and following the orientation. Each vertex in $\mathcal{I}(D)$ has two incoming edges and two outgoing edges, with respect to the orientation; for each vertex, one of the incoming edges will be assigned an even number and the other an odd number, similarly for the outgoing edges. This may be seen from the fact that if we trace along $\mathcal{I}(D)$ from a vertex v, we must encounter other vertices an even number of times before returning to v, since $\mathcal{I}(D)$ is planar.

Thus, the numbering of edges in $\mathcal{I}(D)$ induces a unique numbering of the vertices $0, \ldots, n-1$ by assigning the vertex at which edge $2i$ terminates the number i.

Definition 1.1 We shall refer to the edge numbered $2i$ terminating at vertex i as the **naming edge** for that vertex.

The numbering of the edges of $\mathcal{I}(D)$ also determines a permutation ρ on n elements as follows. At each vertex i the incoming edges are numbered $2i$ and $2j-1$ for some $j \in \{0, \ldots, n-1\}$ where we count edges modulo $2n$. Define $\rho(i) = j$.

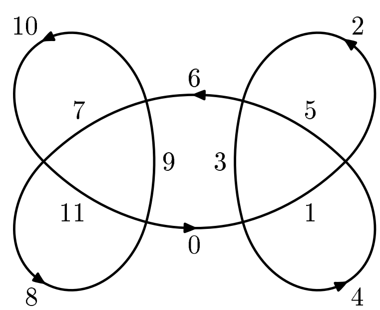

Figure 1.

For example, for the immersion and edge numbering shown in Figure 1, the permutation ρ is

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 0 & 4 \end{matrix}$$

1.1 Type I and Type II crossings

There are two possibilites for the relative numbering of incoming edges at a vertex of $\mathcal{I}(D)$, as shown in Figure 2.

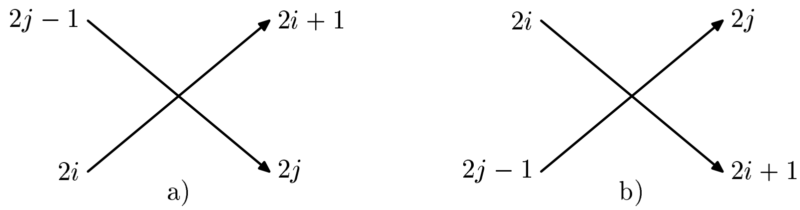

Figure 2.

Definition 1.2 A crossing of the type shown in Figure 2 a) is called a **Type I** crossing and a crossing of the type shown in Figure 2 b) is called a **Type II** crossing.

Definition 1.3 An **immersion code** for the virtual knot diagram D is the permutation ρ determined by a numbering of the edges of $\mathcal{I}(D)$ together with a record of each crossing type. The code is written as a product of cycles describing ρ, with each integer corresponding to a Type I crossing given as a negative number.

Thus the immersion code for Figure 1 is $(-0-2-1-3-5-4)$, where one should note that the minus signs are only an indication of crossing type. Another example of an immersion codes is $(0-3\ 2)(1-4)$, where crossings 0, 1 and 2 are Type II crossings and 3 and 4 are Type I.

1.2 Realizable Immersion Codes

Although patently the immersion code for a knot diagram is not unique, an interesting question is to ask when a given immersion code corresponds to a realizable diagram. This may be answered by noticing that an immersion $\mathcal{I}(D)$ determines a cellular decomposition of S^2 and so by Euler's theorem the number of components of $S^2 - \mathcal{I}(D)$ is $n+2$.

Definition 1.4 For each even numbered edge e in $\mathcal{I}(D)$ there is a sequence of edges $e_0, \ldots, e_k$ with $e = e_0 = e_k$ called the **left turning cycle** obtained by turning left at each crossing we encounter as we trace around $\mathcal{I}(D)$ starting by moving along e following the orientation of $\mathcal{I}(D)$. Similarly we define the **right turning cycle** for e as the corresponding sequence obtained by always turning right. We define left and right turning cycles for odd numbered edges in the same way but require that we start by moving along the edge against the orientation of $\mathcal{I}(D)$.

Clearly every edge in a left (right) turning cycle will determine the same left (right) turning cycle.

Given an immersion code, we are able to determine unambiguously the edge we encounter when turning left or right at a crossing, whether we have arrived following the orientation or not (see Figure 2). Thus we may determine $\mathcal{L}$ the set of distinct left turning cycles, $\mathcal{R}$ the set of distinct right turning cycles, and $c = |\mathcal{L}| + |\mathcal{R}|$.

If each edge appears exactly once in $\mathcal{L}$ and exactly once in $\mathcal{R}$ and if $c = n+2$ then the immersion code is realizable. We may construct a cellular 2-sphere from discs whose boundaries correspond to the turning cycles of $\mathcal{L}$ and $\mathcal{R}$, and whose 1-skeleton is an immersion that yields our given immersion code.

Since we may enumerate permuations of n elements, and may designate crossings as Type I or Type II in only a finite number of ways, we may determine how many realizable immersion codes are possible with n crossings. This has been done by computer search to produce the following table. Permutations where $\rho(i) = i$ or $\rho(i) = (i+1)$ were not considered as these correspond to immersed Reidemeister I configurations.

number of crossings	realizable immersions codes
3	2
4	4
5	12
6	84
7	394
8	1972

2 Labelled Immersion Codes

We may describe a virtual knot diagram D fully by giving its immersion code together with a set of labels that describe each crossing. There are various ways in which one could describe the crossings, hence the teminology, but for now we define just one. We wish to record whether a crossing is real or virtual, and to identify the over-arc in the real case.

Therefore, for real crossings, we assign the label $+$ if the naming edge in $\mathcal{I}(D)$ forms part of the over-arc of the crossing, and the label $-$ if it forms part of the under-arc. If the crossing is virtual, we assign the label $*$.

Definition 2.1 A **labelled immersion code** for a diagram D is an immersion code for D together with a set of labels, one for each crossing. It is written as the immersion code followed by a '/' character, followed in turn by the labels. The labels appear in the order induced on the vertices of $\mathcal{I}(D)$ by the numbering of its edges, as defined in section 1.

Thus, the labelled immersion code for the Kishino knot K3 and numbering shown in Figure 3 is

$$(-0-2-1-3-5-4)/-+*+-*$$

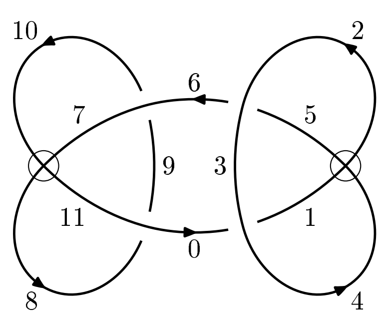

Figure 3.